First order differential equation

Solve the following equation: $3x^2y^2 + 2x^3yy' = 0$

Solution

The given differential equation:

$$3x^2y^2 + 2x^3yy' = 0$$

is an **exact differential equation**. This is because it can be expressed in exact differential form, where there exists a function $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = M(x, y), \quad \frac{\partial F}{\partial y} = N(x, y)$$

and the differential equation is written as:

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

In our case, we can rewrite the equation in the following way:
1. Multiply both sides by dx:

$$3x^2y^2\,dx + 2x^3yy'\,dx = 0$$

2. Observing that $y'\,dx = dy$, we have:

$$3x^2y^2\,dx + 2x^3y\,dy = 0$$

3. Rearranging the terms:

$$(3x^2y^2)\,dx + (2x^3y)\,dy = 0$$

Now, we identify:

$$M(x, y) = 3x^2y^2, \quad N(x, y) = 2x^3y$$

To verify if the equation is exact, we check if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Calculating the partial derivatives:
- $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3x^2y^2) = 6x^2y$
- $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x^3y) = 6x^2y$

Since both partial derivatives are equal, the equation is exact.
Integrate both functions:

$$\int (3x^2y^2)\,dx = x^3y^2 + C_1$$

$$\int (2x^3y)\,dy = x^3y^2 + C_2$$

The final result is:

$$x^3y^2 = C$$